

Fast and flexible interpolation via PUM with applications in population dynamics

Roberto Cavoretto, Alessandra De Rossi, Emma Perracchione

Department of Mathematics "G. Peano", University of Turin - Italy

Abstract. In this paper the Partition of Unity Method (PUM) is efficiently performed using Radial Basis Functions (RBFs) as local approximants. In particular, we present a new space-partitioning data structure extremely useful in applications because of its independence from the problem geometry. Moreover, we study, in the context of wild herbivores in forests, an application of such algorithm. This investigation shows that the ecosystem of the considered natural park is in a very delicate situation, for which the animal population could become extinguished. The determination of the so-called *sensitivity surfaces*, obtained with the new fast and flexible interpolation tool, indicates some possible preventive measures to the park administrators.

1 Introduction

Meshfree methods, first motivated by the fields of geodesy and geophysics, are nowadays popular tools for solving PDEs and interpolation problems. In this paper, given scattered samples, the PUM is performed by blending RBFs as local approximants. This problem is generally considered supposing to interpolate sets of points which are located in suitable domains, such as the square, [2, 5, 10]. Here, focusing on 2D interpolation, we present a tool extremely useful to work in any domain. From this consideration it follows the importance of such a versatile tool in applications. For instance, it can be employed in population dynamics to reconstruct the sensitivity surfaces, analyzed in what follows.

As the name of PUM suggests, an efficient organization of the scattered data among the different PU subdomains turns out to be crucial. To this aim, in literature, techniques known as *kd-trees*, which are not specifically implemented for PUM, have already been designed, [1, 5]. In this paper, starting from the results shown in [2], we propose a versatile partitioning structure, named the *block-based partitioning structure*, and a novel related searching procedure extremely suitable for changes in the problem geometry. Because of the high flexibility of this splitting-data routine, such method can be employed to approximate surfaces defined on irregular domains, as surfaces arising from problems in population dynamics. Specifically, an application in the context of wild herbivores in forests is considered, [6, 8, 9]. We study the case of a natural park located in the Northern Italy. It will be pointed out that it is in a very delicate situation, for which the animal population could become extinguished. The determination of the sensitivity surfaces, via the numerical tool presented here, suggests some treatments to prevent the herbivore extinction.

The paper is organized as follows. In Section 2 we recall theoretical preliminaries on local RBF-based PU approximation, while Section 3 is devoted to the presentation of the block-based partitioning structure. In Section 4 an application in population dynamics of such a stretchy algorithm is considered. Finally Section 5 deals with conclusions pointing out suggestions to prevent the herbivore extinction.

2 Partition of unity method via RBF interpolation

RBF methods are meshfree and work with data given at scattered node points, [2, 5, 10]. We consider the case of N distinct points $\{(x_1, y_1), \dots, (x_N, y_N)\} \subseteq \Omega$, where $\Omega \subseteq \mathbb{R}^2$ is an open and bounded domain. Let $\{f(x_1, y_1), \dots, f(x_N, y_N)\}$ be the associated function values, then the standard RBF interpolation problem consists in finding an interpolant of the form:

$$P(x, y) = \sum_{i=1}^N \lambda_i \varphi(d((x, y), (x_i, y_i))), \quad (1)$$

where $d(\cdot, \cdot)$ is the Euclidean distance, $\lambda_i \in \mathbb{R}$, for $i = 1, \dots, N$ and the function φ is a RBF.

The coefficients $\lambda_1, \dots, \lambda_N$ are determined by enforcing the conditions:

$$P(x_i, y_i) = f(x_i, y_i), \quad i = 1, \dots, N.$$

This leads to a symmetric linear system of equations:

$$A\boldsymbol{\lambda} = \mathbf{f}, \quad (2)$$

where:

$$A_{ij} = \varphi(d((x_i, y_i), (x_j, y_j))), \quad i, j = 1, \dots, N,$$

$$\mathbf{f} = [f(x_1, y_1), \dots, f(x_N, y_N)]^T,$$

and

$$\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_N]^T.$$

In order to describe the PUM, let us consider an open and bounded covering of Ω composed by d subdomains or patches $\{\Omega_j\}_{j=1}^d$ satisfying a point-wise overlap condition. Moreover we also suppose that $\forall (x, y) \in \Omega$, the set $I(x, y) = \{j : (x, y) \in \Omega_j\}$, is such that $\text{card}(I(x, y)) \leq K$, where the constant K is independent from the number of subdomains. Associated with the subdomains we choose a family of compactly supported, nonnegative and continuous functions w_j subordinate to the subdomain Ω_j , such that $\sum_{j=1}^d w_j(x, y) = 1$ on Ω and $\text{supp}(w_j) \subseteq \Omega_j$. Then the global approximant reads as follows:

$$P(x, y) = \sum_{j=1}^d P_j(x, y) w_j(x, y), \quad (x, y) \in \Omega, \quad (3)$$

where P_j defines a local RBF interpolant of the form (1) on each subdomain Ω_j and $w_j : \Omega_j \rightarrow \mathbb{R}$. Thus the global interpolant P is obtained by solving d *small* RBF interpolation problems. Moreover the global PU approximant inherits the interpolation property of the local interpolants, i.e.:

$$P(x_i, y_i) = \sum_{j=1}^d P_j(x_i, y_i) w_j(x_i, y_i) = \sum_{j \in I(x_i, y_i)} f(x_i, y_i) w_j(x_i, y_i) = f(x_i, y_i).$$

3 The block-based partitioning structure

In this section we present our flexible and fast algorithm, built *ad hoc* for bivariate interpolation and extremely useful in applications because of its independence from the problem geometry. We focus on scattered data points arbitrarily distributed in a (priori unknown) domain $\Omega \subseteq \mathbb{R}^2$. Specifically, we approximate the interpolant on the minimal set containing points which can be automatically detected, i.e. at first we compute the convex hull defined by data sites. If the domain is supposed to be known, any generalization is possible and straightforward [7] and consequently this strategy is not restrictive in any sense.

To show our method, used to quickly store points into the different subdomains, we first define the problem geometry. Thus we consider the rectangle \mathcal{R} containing the scattered data:

$$\mathcal{R} = [\min_x, \max_x] \times [\min_y, \max_y],$$

where, as example, with the notation \min_x we identify $\min_i x_i$, $i = 1, \dots, N$. Moreover, a second auxiliary structure, known as *bounding box*, i.e. the square of edge $l_{box} = \max(\max_x, \max_y) - \min(\min_x, \min_y)$ is also considered.

Then, a uniform grid of PU subdomain centres $\mathcal{C}_d = \{(\bar{x}_j, \bar{y}_j), j = 1, \dots, d\}$ is constructed in the convex hull¹. Once the PU structure is built, the problem reduces to solve d RBF interpolation problems. Consequently, a partitioning data structure and a related searching procedure must be employed to efficiently organize points among the d subdomains.

In literature, to this scope the kd-tree partitioning structures are commonly and widely used [1, 5, 10]. Here instead, a novel partitioning structure and a related searching procedure, flexible as kd-trees but with a considerable saving in terms of computational time and cost, is considered. Specifically, after dividing the bounding box auxiliary structure in q^2 squares, named blocks, we store points into such blocks.

The number q of blocks along one side of the bounding box is defined as:

$$q = \left\lceil \frac{l_{box}}{\delta_{PU}} \right\rceil, \quad (4)$$

¹This set is obtained by generating a grid of d_{PU}^2 points on \mathcal{R} , where $d_{PU} = \left\lceil \frac{1}{2} N^{1/2} \right\rceil$. They are then reduced taking only those in Ω .

where $\delta_{PU} = l_{box}\sqrt{2}/d_{PU}$ is the radius of the circular patches. This choice seems to be trivial, but in practice, acting in this way, the search of those points lying in a subdomain whose centre belongs to the k -th block is limited at the k -th neighbourhood, i.e. the k -th block and its eight neighbouring blocks. It leads to a cheap procedure in terms of computational complexity since the searching routine is performed in a constant time, independently from the initial number of nodes.

4 Prevent herbivore extinction via sensitivity surfaces

In this section it will be pointed out how such a flexible procedure, described in the previous sections, can be used to determine the sensitivity surfaces, i.e. surfaces defined in the space of parameters separating the stability regions of two different equilibria.

Let H , G and T represent respectively the herbivores, grass and trees populations of the environment in consideration. The model we consider, [9], is a classical predator with two prey system, in which the resources are consumed following a concave response function, usually called the Beddington-De Angelis function, [4]. Let α and β be the inverse of the herbivores maximal consumption of grass and trees, respectively. Letting r_1 and r_2 denote the grass and trees growth rates, K_1 and K_2 their respective carrying capacities, μ the metabolic rate of herbivores, c and g the half saturation constants, $e \leq 1$ and $f \leq 1$ the conversion factors of food into new herbivore biomass and a and b the daily feeding rates due to grass and trees, respectively, thus the model reads as follows, [8]:

$$\begin{aligned}\dot{H} &= -\mu H + ae \frac{H G}{c + H + \alpha G} + bf \frac{H T}{g + H + \beta T + \alpha G}, \\ \dot{G} &= r_1 G \left(1 - \frac{G}{K_1}\right) - a \frac{H G}{c + H + \alpha G}, \\ \dot{T} &= r_2 T \left(1 - \frac{T}{K_2}\right) - b \frac{H T}{g + H + \beta T + \alpha G}.\end{aligned}\tag{5}$$

All parameters are nonnegative; K_1 , K_2 , c and g are measured in biomass, $e \leq 1$, $f \leq 1$, α and β are pure numbers, μ , r_1 , r_2 , a and b are rates.

For the study of critical points and stability of System (5) refer to [9]. The equilibria, which play a role in this investigation, are the herbivore-free equilibrium $E_1 = (0, K_1, K_2)$ and the coexistence equilibrium point $E^* = (H^*, G^*, T^*)$. The latter can be assessed only via numerical simulations. We test the model considering the natural park of the Dolomiti Bellunesi located in the Northern Italy. To estimate the parameters we refer to [6, 9]. Basing our considerations

on data related to the number of herbivores, the extension of the park and following tables providing the estimation of the annual net primary production of several environments shown in [6], we can fix $\mu = 0.03$, $r_1 = 0.01$, $r_2 = 0.0006$, $\alpha = 0.05^{-1}$, $\beta = 8$, $e = 0.605$, $f = 0.001$, $K_1 = 3469640.64$, $K_2 = 15695993.39$, $c = 101862.16$ and $g = 1001229580.18$. Moreover, also the initial conditions $H(0) = 268.750$, $G(0) = 2313093.76$ and $T(0) = 1046399.56$ are known. With such parameters and initial conditions the model behaves as shown in Figure 1 (left). But we noticed by mean of numerical simulations that the herbivore population level appears to be very sensitive to small perturbations of several parameters and under the high risk of extinction. Figure 1 (right), in the three-dimensional parameter space (μ, e, α) , represents the sensitivity surface separating the stability region of the coexistence equilibrium E^* from the one of the herbivore-free equilibrium E_1 .

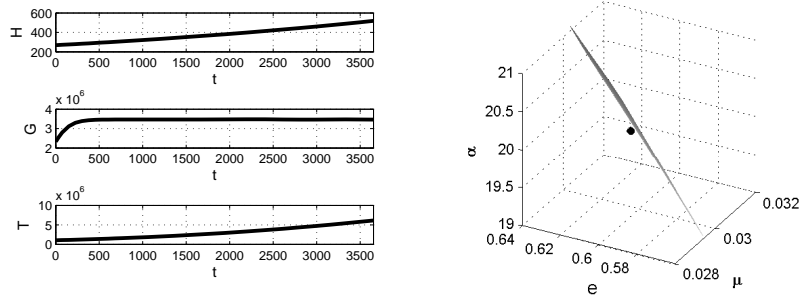


Figure 1: Left: Dolomiti Bellunesi's system evolution over ten years. Right: the surface separating the stability region of the coexistence equilibrium E^* , bottom, from the one of the herbivore-free equilibrium E_1 , top right. The dot represents the actual situation.

To reconstruct such surface we consider as RBF the compactly supported Wendland's C^2 function $\varphi(r) = (1 - \varepsilon r)_+^4 (4\varepsilon r + 1)$, where $\varepsilon \in \mathbb{R}^+$ is the so-called shape parameter and $(\cdot)_+$ denotes the truncated power function. Moreover, since the sensitivity surface is defined on a convex domain, it has been reconstructed via the versatile and flexible partitioning structure described in Section 3. The points describing the surface have been found with a bisection-like routine similar to the one outlined in [3].

We have already pointed out the versatility of the partitioning structure adopted in this paper. Concerning the efficiency, the CPU times, required to interpolate the $N = 107$ *sensitivity points*, with the block-based data structure and, for comparison, with the use of kd-trees² are $t_{bb} = 0.31s$ and $t_{kd} = 5.02s$, respectively.

²We use the only full MATLAB implemented package for kd-trees, written by P. Vemulapalli, available at MATLAB central file exchange <http://www.mathworks.com/matlabcentral/fileexchange/>

5 Final remarks

We presented a new fast and flexible interpolation tool and an application to herbivore population dynamics. From Figure 1 (right) it is easy to verify that the actual value of the herbivore population, represented by the dot, is very close to the separatrix surface. Thus changes in the environmental conditions that might lead even to relatively small parameter perturbations may push it into the region where the herbivores would be doomed.

Moreover, the representation of this sensitivity surface enables us to suggest some treatments in order to possibly prevent the herbivore extinction. Mathematically speaking, the strategy consists in moving away from the sensitivity surface the current value of the parameters μ , e and α . For instance, recalling that μ is the mortality rate, building safety niches during winter surely leads to a decrease of μ . Furthermore, by planting grass more nutritious, the nutrient assimilation coming from grass, i.e. α^{-1} , increases.

References

- [1] S. ARYA, D. M. MOUNT, N. S. NETANYAHU, R. SILVERMAN, A. Y. WU, *An optimal algorithm for approximate nearest neighbor searching in fixed dimensions*, J. ACM **45** (1998), pp. 891–923.
- [2] R. CAVORETTO, A. DE ROSSI, *A meshless interpolation algorithm using a cell-based searching procedure*, Comput. Math. Appl. **67** (2014), pp. 1024–1038.
- [3] R. CAVORETTO, A. DE ROSSI, E. PERRACCHIONE, E. VENTURINO, *Reliable approximation of separatrix manifolds in competition models with safety niches*, Int. J. Comput. Math. **92** (2015), pp. 1826–1837.
- [4] D. DE ANGELIS, R. GOLDSTEIN, R. O’NEILL, *A model for tropic interaction*, Ecol. **56** (1975), pp. 881–892.
- [5] G. E. FASSHAUER, *Meshfree approximation methods with MATLAB*, World Scientific, Singapore, 2007.
- [6] T. FUJIMORI, *Ecological and silvicultural strategies for sustainable forest management*, Elsevier, Amsterdam, Nederland, 2001.
- [7] A. R. H. HERYUDONO, T. A. DRISCOLL, *Radial basis function interpolation on irregular domain through conformal transplantation*, J. Sci. Comput. **44** (2010), pp. 286–300.
- [8] G. SABETTA, E. PERRACCHIONE, E. VENTURINO, *Wild herbivores in forests: four case studies*, to appear in Proc. of BIOMATH14.
- [9] L. TAMBURINO, E. VENTURINO, *Public parks management via mathematical tools*, Int. J. Comp. Math. **89** (2012), pp. 1808–1825.

- [10] H. WENDLAND, *Scattered data approximation*, Cambridge Monogr. Appl. Comput. Math., vol. 17, Cambridge Univ. Press, Cambridge, 2005.